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AN APPROXIMATE SOLUTION FOR AXISYMMETRIC EXTRUSION OF POROUS MATERIAL

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ABSTRACT

The paper concerns with a theoretical solution for axisymmetric extrusion of porous material through a conical die. It is assumed that the material is rigid plastic, obeying Green's yield criterion and its associated flow rule. Coulomb's friction law is adopted at the die surface. Hill's method is used to derive simplified equilibrium equations. Therefore, the solution reduces to ordinary differential equations which are solved numerically. The variation of product porosity with process parameters as well as the distribution of the porosity in the plastic zone is illustrated. The solution is compared to another approximate solution based on a singular yield criterion.
Key word: *axisymmetric extrusion, porous material, Hill's method*

1. INTRODUCTION

The theory of plasticity for porous materials is often adopted to describe the processes of deformation in powder metallurgy [1]. The conventional constitutive equations of this theory are the yield criterion and its associated flow rule. In contrast to the classical metal plasticity [2] where Tresca and Mises yield criteria are usually used, a great variety of yield criteria for porous materials have been proposed and adopted [1, 3 - 7]. It is therefore of interest to understand an effect of the yield criterion on engineering solutions. In the present paper, an approximate solution for the process of extrusion of porous material through a conical die is found. It is assumed that thematerial obeys the yield criterion proposed in [3]. It is a smooth yield criterion and the corresponding yield surface in the space of principal stresses is an ellipsoid. Then, the solution is compared to the solution obtained in [8]. The latter solution is based on a singular yield criterion corresponding to a piece-wise linear surface in the space of principal stresses. In the case of the material with no pores the latter criterion reduces to Tresca yield criterion whereas the yield criterion [3] reduces to Mises yield criterion.

There are several conventional approximate methods used for analysis of metal forming processes. Unfortunately, many of them are not applicable to analysis of forming processes in powder metallurgy. In particular, the slab method deals with the balance of forces. In the case of steady processes the method is not applicable because the distribution of porosity, which influences the stress equations, is unknown in advance. Probably, the most popular method in the case of rigid perfectly/plastic solids is based on the upper bound theorem [2]. However, this method is also not applicable to steady processes in powder metallurgy. The reason for that is that the proof of the upper bound theorem is based on the assumption that the distribution of porosity is known. Obviously, it is unknown in steady processes. Another difficulty with the application of this method is that it is difficult to formulate a physically reasonable friction law compatible with all requirements of the theorem [1]. In the case of axisymmetric extrusion/drawing of rigid perfectly/plastic materials, an approximate method of solution can be based on an analytical solution for material flow through an infinite converging conical channel [9]. For porous materials, a generalization of the solution [9] is only available if friction between the billet and the die is neglected [10, 11]. Therefore, in the present paper, as well as in [8], the approximate method for deriving the equilibrium equation proposed in [12] is adopted. This method has also been successfully applied to other forming processes in powder metallurgy [13-15].

2. STATEMENT OF THE PROBLEM

Geometry of the process is shown in Fig.1. It is natural to adopt a spherical coordinate system $r\varphi\theta$ with its origin at point *O*. In this coordinate system, the die surface is determined by the equation $\varphi = \varphi_0$. It is assumed that the Coulomb friction law is valid on this surface

$$\tau_f = f\left|\sigma_n\right| \qquad \text{at} \qquad \varphi = \varphi_0 \tag{1}$$

where τ_f is the friction stress, *f* is the friction coefficient, and σ_n is the normal stress acting on the friction surface. It is assumed that the velocity of the punch and the porosity at the entrance are given, *U* and ϑ_0, respectively. No force is applied at the exit end. The force applied to the punch, *P*, should be found from the solution. There are also natural boundary conditions at the axis of symmetry. The yield criterion proposed in [3] can be written in the form

$$\frac{\sigma^2}{p_s^2} + \frac{\tau^2}{\tau_s^2} = 1 \tag{2}$$

Here σ is the hydrostatic stress, τ is the second invariant of the stress tensor, p_s is the yield stress at the hydrostatic compression, and τ_s is the shear yield stress. Both p_s and τ_s depend on the porosity, ϑ. In particular, $p_s \to \infty$ and $\tau_s \to k$ as $\vartheta \to 0$, where k is the shear yield stress of the matrix material. In the case of axisymmetric deformation, in terms of the stress components in the spherical coordinates, σ and τ are expressed as

$$\sigma = \frac{\sigma_{rr} + \sigma_{\varphi\varphi} + \sigma_{\theta\theta}}{3}, \qquad \tau = \sqrt{\frac{1}{2}\left[\left(\sigma_{rr} - \sigma\right)^2 + \left(\sigma_{\varphi\varphi} - \sigma\right)^2 + \left(\sigma_{\theta\theta} - \sigma\right)^2 + 2\sigma_{r\varphi}^2\right]} \tag{3}$$

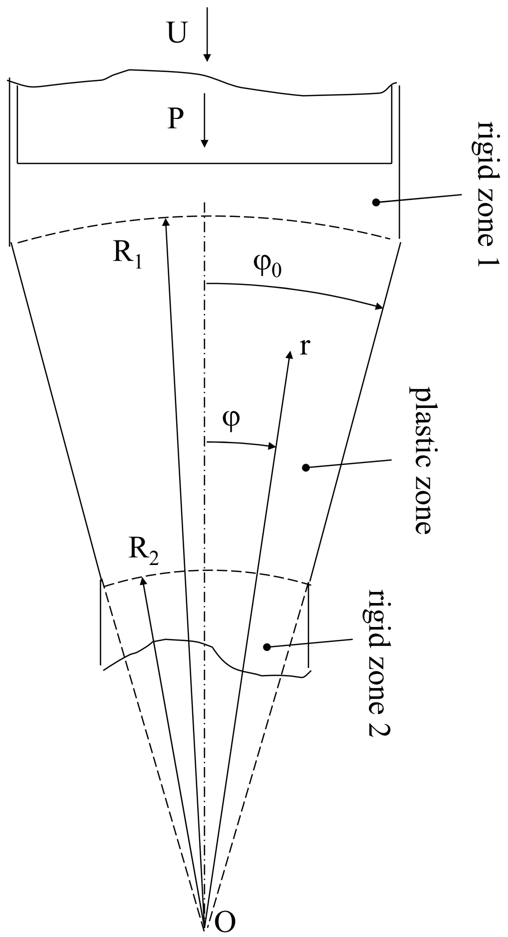

Figure 1 - Geometry of the process.

The associated flow rule is given by [1]

$$\xi_{rr} = \lambda\left(\frac{2}{3}\frac{\sigma}{p_s^2} + \frac{\tau_{rr}}{\tau_s^2}\right), \qquad \xi_{\varphi\varphi} = \lambda\left(\frac{2}{3}\frac{\sigma}{p_s^2} + \frac{\tau_{\varphi\varphi}}{\tau_s^2}\right),$$
$$\xi_{\theta\theta} = \lambda\left(\frac{2}{3}\frac{\sigma}{p_s^2} + \frac{\tau_{\theta\theta}}{\tau_s^2}\right), \qquad \xi_{r\varphi} = \lambda\frac{\tau_{r\varphi}}{\tau_s^2} \tag{4}$$

where ξ_{rr}, $\xi_{\varphi\varphi}$, $\xi_{\theta\theta}$ and $\xi_{r\varphi}$ are the components of the strain rate tensor in the spherical coordinates (the other components vanish because of axial symmetry), $\tau_{rr} = \sigma_{rr} - \sigma$, $\tau_{\varphi\varphi} = \sigma_{\varphi\varphi} - \sigma$, $\tau_{\theta\theta} = \sigma_{\theta\theta} - \sigma$, $\tau_{r\varphi} \equiv \sigma_{r\varphi}$, and λ is a non-negative scalar multiplier. The continuity equation is written in the form

$$\frac{d\vartheta}{dt} = (1-\vartheta)\left(\xi_{rr} + \xi_{\varphi\varphi} + \xi_{\theta\theta}\right) \tag{5}$$

where d/dt is the material derivative with respect to time.

3. KINEMATICS OF THE PROCESS

In order to apply the method proposed in [12], it is necessary to select an appropriate velocity field. The simplest velocity field in the plastic zone (Fig.1) is

$$u_r = u(r), \quad u_\varphi = 0, \quad u_\theta = 0 \tag{6}$$

where $u(r)$ is an arbitrary function of *r*. Obviously, the velocity field (6) satisfies the velocity boundary conditions at $\varphi = \varphi_0$ and $\varphi = 0$ where u_φ must vanish. Using (6) it is possible to find the strain rate components

$$\xi_{rr} = \frac{du}{dr}, \quad \xi_{\varphi\varphi} = \frac{u}{r}, \quad \xi_{\theta\theta} = \frac{u}{r}, \quad \xi_{r\varphi} = 0 \tag{7}$$

Since the flow is stationary, the continuity equation (5), with the use of (6) and (7), reduces to

$$\frac{u}{(1-\vartheta)}\frac{d(1-\vartheta)}{dr} + \frac{du}{dr} + 2\frac{u}{r} = 0 \tag{8}$$

The general solution to this equation has the form

$$u = -\frac{C}{(1-\vartheta)r^2}, \quad C > 0 \tag{9}$$

The constant of integration, *C*, can be found from the condition that the material flux through the surface $r = R_1$ (Fig. 1) is the same on each of its sides. In particular, the value of *u* at $r = R_1$ is determined from

$$u(R_1) = -U(1+\cos\varphi_0)/2 \tag{10}$$

Since $\cos\varphi_0 \approx 1$ for small φ_0 and the assumptions made can only be appropriate for such small values of φ_0 it is possible to get from (9) and (10) that

$$C = (1-\vartheta_0)R_1^2 U \tag{11}$$

4. STRESS EQUATIONS

It follows from (4) and (7) that

$$\tau_{r\varphi} = 0 \tag{12}$$

and $\tau_{\varphi\varphi} = \tau_{\theta\theta}$. Combining the latter equation and the identity $\tau_{rr} + \tau_{\varphi\varphi} + \tau_{\theta\theta} = 0$ gives

$$\tau_{\varphi\varphi} = \tau_{\theta\theta} = -\tau_{rr}/2 \tag{13}$$

Substituting (12) and (13) into the yield condition (2) and using (3) leads to

$$\frac{\sigma^2}{p_s^2} + \frac{3\tau_{\varphi\varphi}^2}{\tau_s^2} = 1 \tag{14}$$

Equation (14) is satisfied by the substitution

$$\sigma = -p_s \cos\gamma, \qquad \tau_{\varphi\varphi} = -\tau_s \sin\gamma/\sqrt{3} \tag{15}$$

It is reasonable to assume that $\sigma < 0$ and $\tau_{rr} > \tau_{\varphi\varphi}$. Then, it follows from (13) and (15) that

$$0 \le \gamma \le \pi/2 \tag{16}$$

The approximate equilibrium equation derived according to the method [12] depends on the velocity field chosen and stress boundary conditions. Since the velocity field chosen in [8] coincides with (6) and the friction law selected in [8] coincides with (1) where $\sigma_n = \sigma_{\varphi\varphi}$, the equilibrium equation found in this work is appropriate for the problem under consideration. The equation has the following form [8]

$$r\, d\sigma_{rr}/dr + 2\sigma_{rr} - 2m\sigma_{\varphi\varphi} = 0 \tag{17}$$

where $m = 1 + (1/2) f \cot(\varphi_0/2)$. The stresses involved in (17) can be expressed in terms of γ with the use (15)

$$\sigma_{rr} = -p_s \cos\gamma + 2\tau_s \sin\gamma/\sqrt{3}, \quad \sigma_{\varphi\varphi} = -p_s \cos\gamma - \tau_s \sin\gamma/\sqrt{3} \tag{18}$$

Substituting (18) into (17) gives

$$\begin{aligned} & r\left(\frac{2}{\sqrt{3}}\sin\gamma \frac{d\tau_s}{d\vartheta} - \cos\gamma \frac{dp_s}{d\vartheta}\right)\frac{d\vartheta}{dr} + r\left(p_s \sin\gamma + \frac{2}{\sqrt{3}}\tau_s \cos\gamma\right)\frac{d\gamma}{dr} + \\ & + 2(m-1) p_s \cos\gamma + \frac{2}{\sqrt{3}}(m+2)\tau_s \sin\gamma = 0 \end{aligned} \tag{19}$$

5. ASSOCIATED FLOW RULE

It follows from (7) that $\xi_{\varphi\varphi} = \xi_{\theta\theta}$ and $\xi_{r\varphi} = 0$. Therefore, it is sufficient to consider the first two equations of the associated flow rule (4). Excluding λ in (4) and using (13) it is possible to arrive at the equation

$$\frac{\xi_{rr}}{\xi_{\varphi\varphi}} = \frac{2\tau_s^2\sigma - 6p_s^2\tau_{\varphi\varphi}}{2\tau_s^2\sigma + 3p_s^2\tau_{\varphi\varphi}} \tag{20}$$

Taking into account (7) and (15) equation (20) can be rewritten in the form

$$\frac{du}{dr} = \frac{2u}{r}\left(\frac{\tau_s\cos\gamma - \sqrt{3}p_s\sin\gamma}{2\tau_s\cos\gamma + \sqrt{3}p_s\sin\gamma}\right)$$

Using (9) this equation can be transformed to

$$r\frac{d\vartheta}{dr} = \frac{6(1-\vartheta)\tau_s\cos\gamma}{\left(2\tau_s\cos\gamma + \sqrt{3}p_s\sin\gamma\right)} \tag{21}$$

Since τ_s and p_s are supposed to be given functions of ϑ, equations (19) and (21) form the closed form system with respect to ϑ and γ. This system should be solved numerically.

6. NUMERICAL SOLUTION

The dependence of τ_s and p_s on the porosity can be assumed in the form [1]

$$\tau_s = k(1-\vartheta)^{3/2} \quad \text{and} \quad p_s = \frac{2k(1-\vartheta)^2}{\sqrt{3}\sqrt{\vartheta}} \tag{22}$$

Using these dependencies it is possible to find the derivatives of τ_s and p_s involved in equation (19)

$$\frac{d\tau_s}{d\vartheta} = -\frac{3k}{2}\sqrt{1-\vartheta} \quad \text{and} \quad \frac{dp_s}{d\vartheta} = -\frac{k}{\sqrt{3}}\frac{(1-\vartheta)(1+3\vartheta)}{\vartheta^{3/2}} \tag{23}$$

Equations (19) and (21) can be transformed to

$$\left(p_s \sin\gamma + \frac{2}{\sqrt{3}}\tau_s \cos\gamma \right)\frac{d\gamma}{d\vartheta} = \left(\cos\gamma \frac{dp_s}{d\vartheta} - \frac{2}{\sqrt{3}}\sin\gamma \frac{d\tau_s}{d\vartheta} \right) - \\ - \frac{\left[2(m-1) p_s \cos\gamma + \frac{2}{\sqrt{3}}(m+2)\tau_s \sin\gamma \right]\left(2\tau_s \cos\gamma + \sqrt{3} p_s \sin\gamma \right)}{6(1-\vartheta)\tau_s \cos\gamma} \quad 24)$$

Using (22) and (23) the coefficients of this equation can be represented as functions of ϑ and γ. Since $\sigma_{rr} = 0$ at $r = R_2$ (Fig.1), the first of relations (18) gives

$$-\sqrt{3} p_s(\vartheta_f)\cos\gamma_2 + 2\tau_s(\vartheta_f)\sin\gamma_2 = 0 \quad (25)$$

at $r = R_2$. Here ϑ_f is the porosity of the final product and γ_2 is the value of γ at $r = R_2$. Equation (25) provides the boundary condition to equation (24). Once equation (24) has been solved, the distribution of the porosity in the plastic zone can be found from (21) by integration in implicit form

$$\frac{r}{R_2} = \exp\left[\frac{1}{6}\int_{\vartheta_f}^{\vartheta} \frac{\left(2\tau_s \cos\gamma + \sqrt{3} p_s \sin\gamma \right)}{(1-\vartheta)\tau_s \cos\gamma} d\vartheta \right] \quad (26)$$

Note the in this equation γ is the known function of ϑ due to the solution to equation (24). Since $\vartheta = \vartheta_0$ at $r = R_1$, it follows from (26) that

$$\frac{R_1}{R_2} = \exp\left[\frac{1}{6}\int_{\vartheta_f}^{\vartheta_0} \frac{\left(2\tau_s \cos\gamma + \sqrt{3} p_s \sin\gamma \right)}{(1-\vartheta)\tau_s \cos\gamma} d\vartheta \right] \quad (27)$$

If ϑ_f is prescribed, equation (27) determines ϑ_0. On the other hand, if ϑ_0 is prescribed, an iterative procedure should be used to satisfy equation (27) and thus find ϑ_f. In either case the extrusion ration, R_1/R_2, is supposed to be given.

The yield criterion (2) imposes a restriction on the maximum possible value of the shear stress supported by the material. For instant, the material supports no shear stress if $|\sigma| = p_s$. In the case of exact solutions, this restriction is naturally accounted for in the solutions. Since the present solution is approximate, it is necessary to take special measures to exclude the state of stress where the friction stress is larger than the maximum possible shear stress permissible by the yield criterion. In particular, it follows from (2) that the maximum possible shear stress is

$$\tau_{\max} = \tau_s \sqrt{1 - (\sigma/p_s)^2} \quad (28)$$

Then, combining (1) and (28), with the use of (15) and (18), yields

$$f\left(p_s\cos\gamma+\frac{1}{\sqrt{3}}\tau_s\sin\gamma\right)\le\tau_s\sin\gamma \tag{29}$$

Substituting (22) and the solution to (24) into (29) gives the minimum possible value of the initial porosity at any given value of the final porosity. The variation of such minimum value of ϑ_0 with the final porosity is illustrated in Figs.2 and 3 at different values of the friction coefficient and $\varphi_0 = 5^0$ and $\varphi_0 = 10^0$, respectively. To get the solution for smaller values of ϑ_0, it is necessary to modify the friction law (1). In the present paper, the range of parameters is chosen such that the inequality (29) is satisfied.

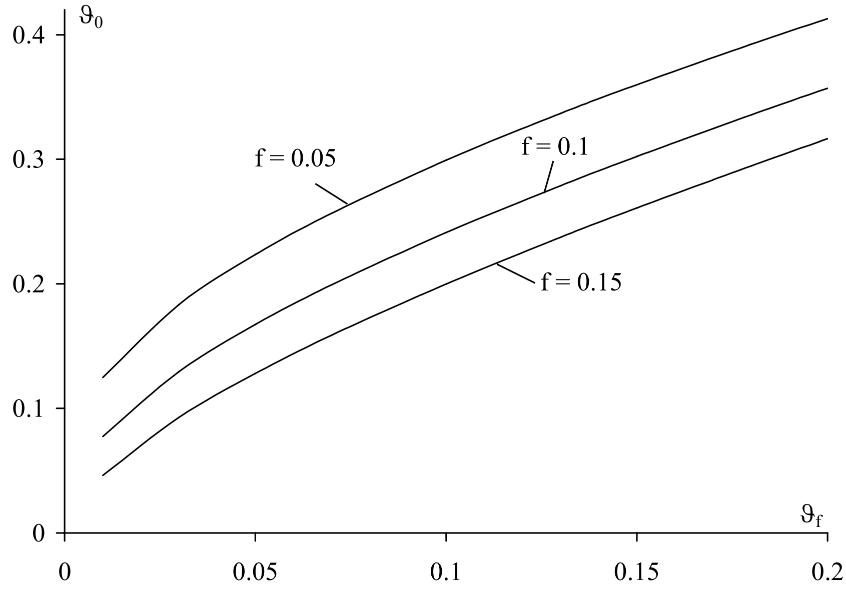

Figure 2 - Variation of the minimum initial porosity with the final porosity and friction coefficient at $\varphi_0 = 5^0$.

The numerical solution to equations (24) and (26) is illustrated in Figs 5 to 7 where the dependence of the initial porosity, ϑ_0, necessary to get the prescribed final porosity, ϑ_f, on the extrusion ratio, $\ln(R_1/R_2)$, is shown at different values of the friction coefficient, *f*, and the die angle, φ_0. To calculate these curves, equation (27) has been used. The right ends of the curves depicted in Figs 5 to 7 correspond to the case where the equality is satisfied in (29).

It is of interest to compare the solution found and the solution found in [8] with the use of a singular yield criterion. In the space of principal stresses, this criterion is represented by a right pyramid at $\sigma \le 0$ and is symmetric relative to the plane $\sigma = 0$. A complete description of this yield criterion and its associated flow rule is given in [1]. Because of special properties of these equations, the kinematics of the process, including the distribution of the porosity, is determined in [8] independently of the stress equations. Therefore, this part of the solution [8] is independent of the friction coefficient and the die angle, which are only involved in the stress equations. The comparison of the two solutions is illustrated in Figs. 8 to 10 at different values of the friction coefficient and die angle. In each of the considered cases corresponding to the present solution, the curves obtained at $\varphi_0 = 10^0$, $f = 0.1$ and $\varphi_0 = 5^0$, $f = 0.05$ almost coincide (the difference is not visible on the diagrams). This shows that the friction coefficient and die angle have a similar effect on the porosity distribution. The curve corresponding to the solution [8] is shown by a dashed line on each graph (there is one curve on each graph because the solution is independent of φ_0 and *f*). It is seen from Figs. 8 to 10 that the difference between solutions based on the two yield criteria can be quite large.

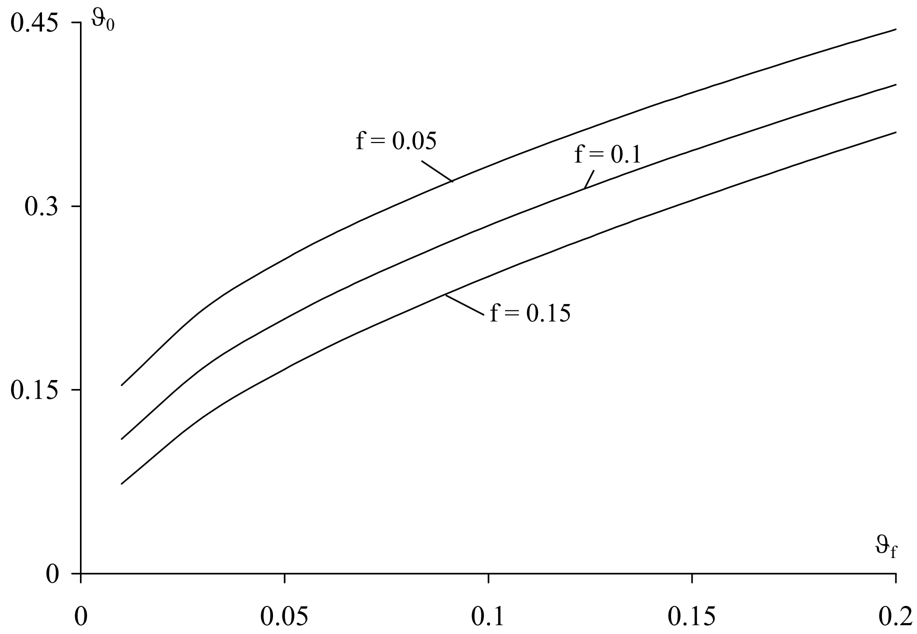

Figure 3 - Variation of the minimum initial porosity with the final porosity and friction coefficient at $\varphi_0 = 10^0$.

7. CONCLUSIONS

A new approximate solution for axisymmetric extrusion of porous material through a conical die has been found. The main goal of the research has been to show an effect of different yield criteria on the distribution of porosity in the plastic zone, including the value of porosity in the final product. To this end, the assumptions made in [8] have been also adopted in the present solution. The comparison of the two solutions illustrated in Figs 8 to 10 shows that the yield criterion can have a significant effect on the final result.

The velocity distribution in the plastic zone can be found with the use of the solution to equations (24) and (26) combined with (9) and (11). The solution to equations (24) and (26) can also be combined with (18) to find the distribution of the stress components in the plastic zone including the surface $r = R_1$ (Fig.1). Then, the force required to push the material through the die can be found by integrating the radial stress at $r = R_1$ over this surface.

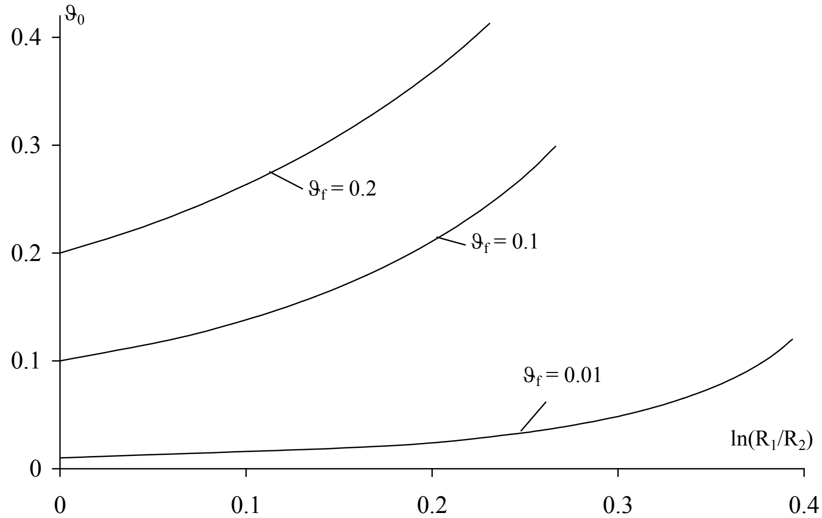

Figure 4 - Variation of the initial porosity ϑ_0 required to get the final porosity ϑ_f with the extrusion ratio at $\varphi_0 = 5^0$ and $f = 0.05$.

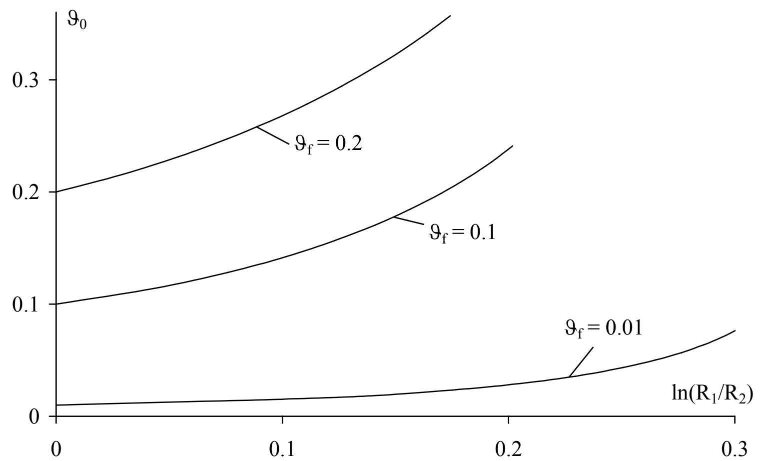

Figure 5 - Variation of the initial porosity ϑ_0 required to get the final porosity ϑ_f with the extrusion ratio at $\varphi_0 = 5^0$ and $f = 0.1$.

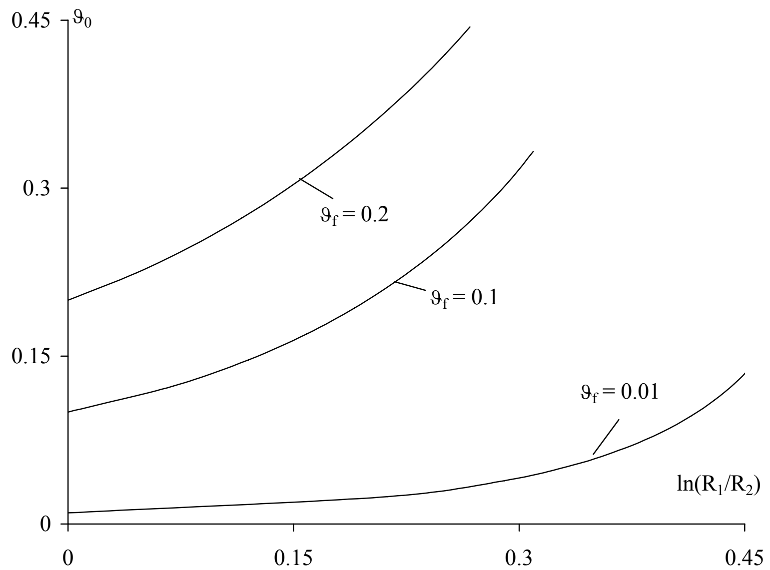

Figure 6 - Variation of the initial porosity ϑ_0 required to get the final porosity ϑ_f with the extrusion ratio at $\varphi_0 = 10^0$ and $f = 0.05$.

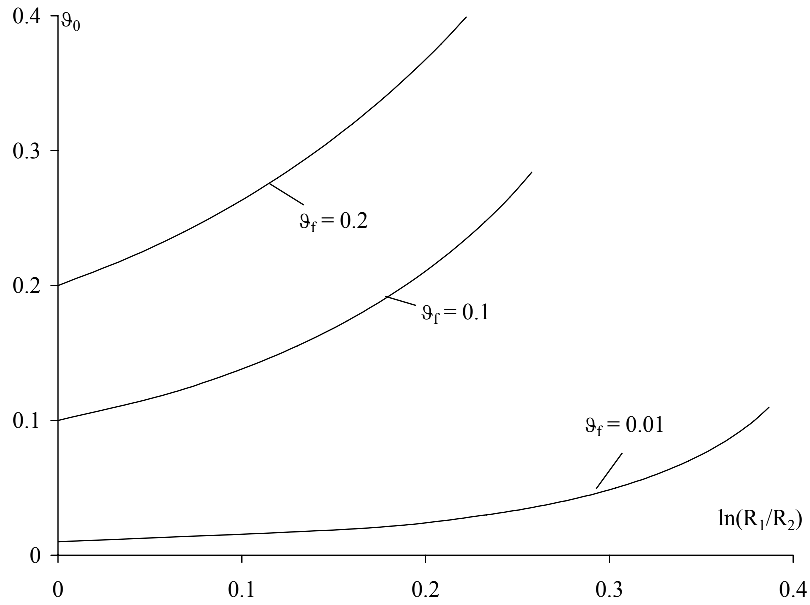

Figure 7 - Variation of the initial porosity ϑ_0 required to get the final porosity ϑ_f with the extrusion ratio at $\varphi_0 = 10^0$ and $f = 0.1$.

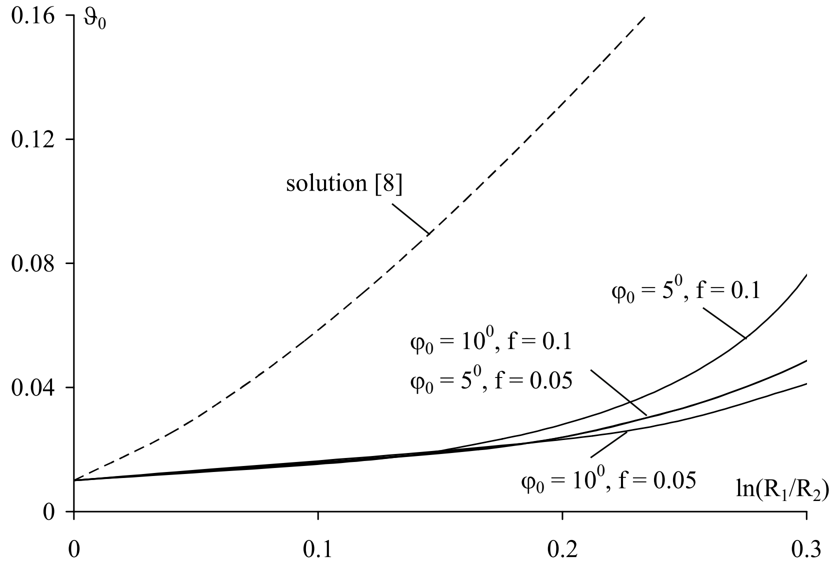

Figure 8 - Comparison of the present solution and solution obtained in [8] at $\vartheta_f = 0.01$.

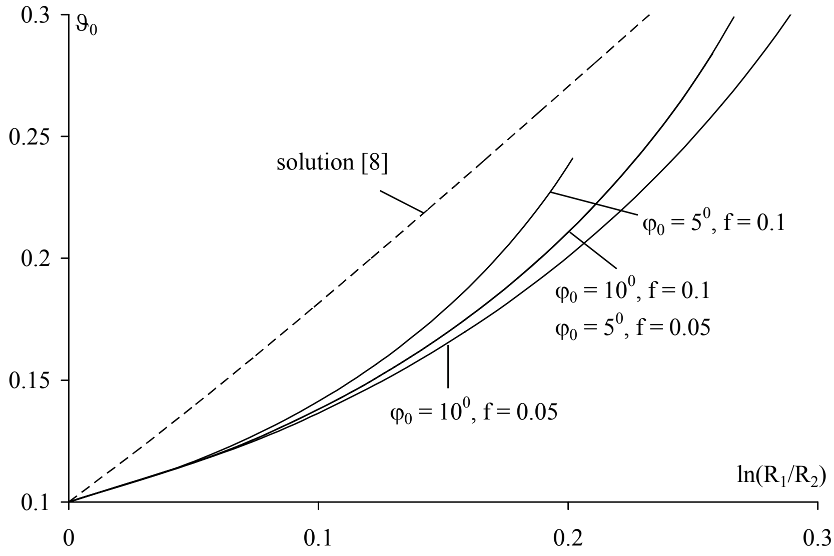

Figure 9 - Comparison of the present solution and solution obtained in [8] at $\vartheta_f = 0.1$.

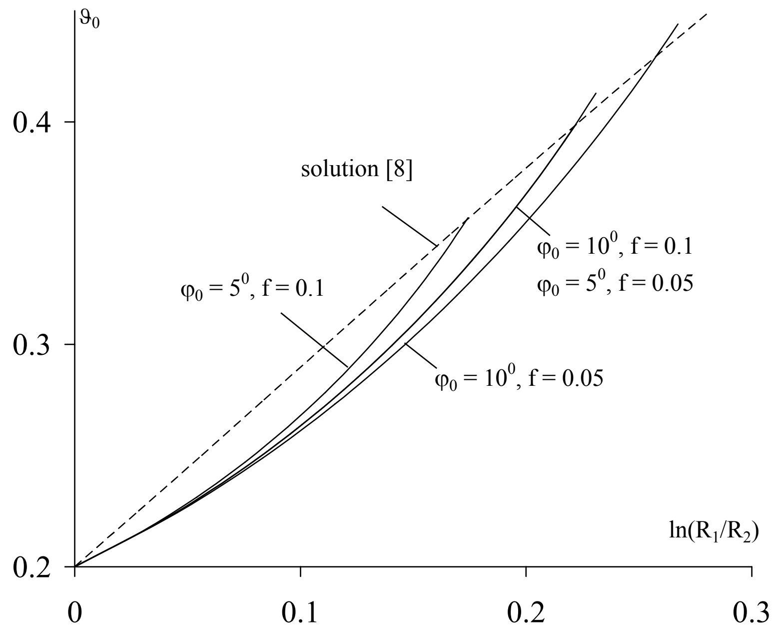

Figure 10 - Comparison of the present solution and solution obtained in [8] at $\vartheta_f = 0.2$.

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PRIBLIŽNO REŠENJE AKSIJALNO SIMETRIČNOG ISTISKIVANJA POROZNIH MATERIJALA

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REZIME

Rad se bavi iznalaženjem teoretnog rešenja aksijalno simetričnog istiskivanja poroznih materijala kroz konusno konvergenti kanal. Predpostavlja se da je materijal kruto plastičan i da se ponaša po Green-ovom kriterijumu plastičnosti. Usvojen je Coulombov zakon trenja na graničnim površinama. Za dobijanje pojednostavljenih jednačina ravnoteže korišćena je Hillova teorija. Zbog toga je rešenje ograničeno na obične diferencijalne jednačine koje su rešene numerički. Ilustrovana je promena poroznosti proizvoda u vezi sa ostalim parametrima procesa kao i raspored poroznosti u plastičnoj zoni.
Dobijena rešenja su upoređene sa drugim približnim rešenjima baziranim na singularnom uslovu plastičnosti.